

THE CENTER OF A KUMJIAN-PASK ALGEBRA

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ABSTRACT. The Kumjian-Pask algebras are path algebras associated to higher-rank graphs, and generalize the Leavitt path algebras. We study the center of simple Kumjian-Pask algebras and characterize commutative Kumjian-Pask algebras.

1. INTRODUCTION

Let E be a directed graph E and let \mathbb{F} be a field. The Leavitt path algebras $L_{\mathbb{F}}(E)$ of E over \mathbb{F} were first introduced in [1] and [2], and have been widely studied since then. Many of the properties of Leavitt path algebras can be inferred from properties of the graph, and for this reason provide a convenient way to construct examples of algebras with a particular set of attributes. The Leavitt path algebras are the algebraic analogues of the graph C^* -algebras associated to E . In [10], Tomforde constructed an analogous Leavitt path algebra $L_R(E)$ over a commutative ring R with 1, and introduced more techniques from the graph C^* -algebra setting to study it.

In [3], Aranda Pino, Clark, an Huef and Raeburn generalized Tomforde's construction and associated to a higher-rank graph Λ a graded algebra $KP_R(\Lambda)$ called the *Kumjian-Pask algebra*. Example 7.1 of [3] shows that even the class of commutative Kumjian-Pask algebras over a field is strictly larger than the class of Leavitt path algebras over that field.

In this paper we study the center of Kumjian-Pask algebras. In §3 we work over \mathbb{C} and show how the embedding of $KP_{\mathbb{C}}(\Lambda)$ in the C^* -algebra $C^*(\Lambda)$ can be used together with the Dauns-Hofmann theorem to deduce that the center of a simple Kumjian-Pask algebra is either $\{0\}$ or isomorphic to \mathbb{C} .

More generally, it follows from Theorem 4.7, that the center of a “basically simple” (see page 7) Kumjian-Pask algebra $KP_R(\Lambda)$ is either zero or is isomorphic to the underlying ring R . Thus our Theorem 4.7 generalizes the analogous theorem for Leavitt path algebras over a field [4, Theorem 4.2], but our proof techniques are very different and more informative. Indeed, the Kumjian-Pask algebra is basically simple if and only if the graph Λ is cofinal and aperiodic, and our proofs show explicitly which of these graph properties are needed to infer various properties of elements in the center.

In Proposition 5.3 we show that a Kumjian-Pask algebra of a k -graph Λ is commutative if and only if it is a direct sum of rings of Laurant polynomials in k -indeterminates, if and only if Λ is a disjoint union of copies of the category \mathbb{N}^k . This generalizes Proposition 2.7 of [4].

2. PRELIMINARIES

We view \mathbb{N}^k as a category with one object and composition given by addition. We call a countable category $\Lambda = (\Lambda^0, \Lambda, r, s)$ a k -graph if there exists a functor $d : \Lambda \rightarrow \mathbb{N}^k$ with the *unique factorization property*: for all $\lambda \in \Lambda$, $d(\lambda) = m + n$ implies there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m, d(\nu) = n$ and $\lambda = \mu\nu$. Using the unique

factorization property, we identify the set of objects Λ^0 with the set of morphisms of degree 0. Then, for $n \in \mathbb{N}^k$, we set $\Lambda^n := d^{-1}(n)$, and call Λ^n the paths of shape n in Λ and Λ^0 the vertices of Λ . A path $\lambda \in \Lambda$ is *closed* if $r(\lambda) = s(\lambda)$.

For $V, W \subset \Lambda^0$, we set $V\Lambda := \{\lambda \in \Lambda : r(\lambda) \in V\}$, $\Lambda W := \{\lambda \in \Lambda : s(\lambda) \in W\}$ and $V\Lambda W := V\Lambda \cap \Lambda W$; the sets $V\Lambda^n$, $\Lambda^n W$ and $V\Lambda^n W$ are defined similarly. For simplicity we write $v\Lambda$ for $\{v\}\Lambda$.

A k -graph Λ is *row-finite* if $|v\Lambda^n| < \infty$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ and has *no sources* if $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. We assume throughout that Λ is a row-finite k -graph with no sources.

Following [9, Lemma 3.2(iv)], we say that a k -graph Λ is *aperiodic* if for every $v \in \Lambda^0$ and $m \neq n \in \mathbb{N}^k$ there exists $\lambda \in v\Lambda$ such that $d(\lambda) \geq m \vee n$ and

$$\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n)).$$

This formulation of aperiodicity is equivalent to the original one from [7, Definition 4.3] when Λ is a row-finite graph with no sources, but is often more convenient since it only involves finite paths.

Let $\Omega_k := \{(m, n) \in \mathbb{N}^k : m \leq n\}$. As in [7, Definition 2.1] we define an *infinite path* in Λ to be a degree-preserving functor $x : \Omega_k \rightarrow \Lambda$, and denote the set of infinite paths by Λ^∞ . As in [7, Definition 4.1] we say Λ is *cofinal* if for every infinite path x and every vertex v there exists $m \in \mathbb{N}^k$ such that $v\Lambda x(m) \neq \emptyset$.

For each $\lambda \in \Lambda$ we introduce a *ghost path* λ^* ; for $v \in \Lambda^0$ we take $v^* = v$. We write $G(\Lambda)$ for the set of ghost paths and $G(\Lambda^{\neq 0})$ if we exclude the vertices.

Let R be a commutative ring with 1. Following [3, Definition 3.1], a *Kumjian-Pask Λ -family* (P, S) in an R -algebra A consists of functions $P : \Lambda^0 \rightarrow A$ and $S : \Lambda^{\neq 0} \cup G(\Lambda^{\neq 0}) \rightarrow A$ such that

- (KP1) $\{P_v : v \in \Lambda^0\}$ is a set of mutually orthogonal idempotents;
- (KP2) for $\lambda, \mu \in \Lambda^{\neq 0}$ with $r(\mu) = s(\lambda)$,

$$S_\lambda S_\mu = S_{\lambda\mu}, \quad S_{\mu^*} S_{\lambda^*} = S_{(\lambda\mu)^*}, \quad P_{r(\lambda)} S_\lambda = S_\lambda = S_\lambda P_{s(\lambda)}, \quad P_{s(\lambda)} S_{\lambda^*} = S_{\lambda^*} = S_{\lambda^*} P_{r(\lambda)};$$

- (KP3) for all $\lambda, \mu \in \Lambda^{\neq 0}$ with $d(\lambda) = d(\mu)$, we have $S_{\lambda^*} S_\mu = \delta_{\lambda, \mu} P_{s(\lambda)}$;

- (KP4) for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k \setminus \{0\}$, we have $P_v = \sum_{\lambda \in v\Lambda^n} S_\lambda S_{\lambda^*}$.

By [3, Theorem 3.4] there is an R -algebra $\text{KP}_R(\Lambda)$, generated by a nonzero Kumjian-Pask Λ -family (p, s) , with the following universal property: whenever (Q, T) is a Kumjian-Pask Λ -family in an R -algebra A , then there is a unique R -algebra homomorphism $\pi_{Q,T} : \text{KP}_R(\Lambda) \rightarrow A$ such that

$$\pi_{Q,T}(p_v) = Q_v, \quad \pi_{Q,T}(s_\lambda) = T_\lambda \text{ and } \pi_{Q,T}(s_{\mu^*}) = T_{\mu^*} \text{ for } v \in \Lambda^0 \text{ and } \lambda, \mu \in \Lambda^{\neq 0}.$$

Also by Theorem 3.4 of [3], the subgroups

$$\text{KP}_R(\Lambda)_n := \text{span}_R\{s_\lambda s_{\mu^*} : \lambda, \mu \in \Lambda \text{ and } d(\lambda) - d(\mu) = n\} \quad (n \in \mathbb{Z}^k)$$

give a \mathbb{Z}^k -grading of $\text{KP}_R(\Lambda)$. Let S be a \mathbb{Z}^k -graded ring; then by the graded-uniqueness theorem [3, Theorem 4.1], a graded homomorphism $\pi : \text{KP}_R(\Lambda) \rightarrow S$ such that $\pi(rp_v) \neq 0$ for nonzero $r \in R$ is injective.

We will often write elements $a \in \text{KP}_R(\Lambda) \setminus \{0\}$ in the *normal form* of [3, Lemma 4.2]: there exists $m \in \mathbb{N}^k$ and a finite $F \subset \Lambda \times \Lambda^m$ such that $a = \sum_{(\alpha, \beta) \in F} r_{\alpha, \beta} s_\alpha s_{\beta^*}$ where $r_{\alpha, \beta} \in R \setminus \{0\}$ and $s(\alpha) = s(\beta)$.

3. MOTIVATION

When A is a simple C^* -algebra (over \mathbb{C} , of course), it follows from the Dauns-Hofmann Theorem (see, for example, [8, Theorem A.34]) that the center $Z(A)$ of A is isomorphic to \mathbb{C} if A has an identity and is $\{0\}$ otherwise. Let Λ be a row-finite k -graph without sources. In this short section we deduce that the center of a simple Kumjian-Pask algebra $\text{KP}_{\mathbb{C}}(\Lambda)$ is either isomorphic to \mathbb{C} or is $\{0\}$.

Lemma 3.1. *Suppose A is a simple C^* -algebra. If A has an identity, then $z \mapsto z1_A$ is an isomorphism of \mathbb{C} onto the center $Z(A)$ of A . If A has no identity, then $Z(A) = \{0\}$.*

Proof. Since A is simple, $\text{Prim } A = \{\star\}$, and $f \mapsto f(\star)$ is an isomorphism of $C_b(\text{Prim } A)$ onto \mathbb{C} . By the Dauns-Hofmann Theorem, $C_b(\text{Prim } A)$ is isomorphic to the center $Z(M(A))$ of the multiplier algebra $M(A)$ of A . Putting the two isomorphisms together gives an isomorphism $z \mapsto z1_{M(A)}$ of \mathbb{C} onto $Z(M(A))$.

Now suppose that A has an identity. Then $M(A) = A$, and it follows from the first paragraph that $Z(A)$ is isomorphic to \mathbb{C} .

Next suppose that A does not have an identity. Let $a \in Z(A)$, and let u_λ be an approximate identity in A and $m \in M(A)$. Then $ma = \lim(mu_\lambda)a = a \lim(mu_\lambda) = am$. Thus $Z(A) \subset Z(M(A))$. Now $Z(A) \subset Z(M(A)) \cap A = \{z1_{M(A)} : z \in \mathbb{C}\} \cap A = \{0\}$. \square

Lemma 3.2. *Let D be a dense subalgebra of a C^* -algebra A . Then $Z(A) \cap D = Z(D)$.*

Proof. Trivially, $Z(A) \cap D \subset Z(D)$. To see the reverse inclusion, let $a \in Z(D)$. Let $b \in A$ and choose $\{d_\lambda\} \subset D$ such that $d_\lambda \rightarrow b$. Then $ba = \lim_\lambda d_\lambda a = \lim_\lambda a d_\lambda = ab$. Now $a \in Z(A) \cap Z(D) \subset Z(A) \cap D$, and hence $Z(A) \cap D = Z(D)$. \square

By [3, Theorem 6.1], $\text{KP}_{\mathbb{C}}(\Lambda)$ is simple if and only if Λ is cofinal and aperiodic, so in the next corollary $\text{KP}_{\mathbb{C}}(\Lambda)$ is simple. Also, $\text{KP}_{\mathbb{C}}(\Lambda)$ has an identity if and only if Λ^0 is finite (see Lemma 4.6 below).

Corollary 3.3. *Suppose that Λ is a row-finite, cofinal, aperiodic k -graph with no sources. If Λ^0 is finite, then $z \mapsto z1_{\text{KP}_{\mathbb{C}}(\Lambda)}$ is an isomorphism of \mathbb{C} onto the center $Z(\text{KP}_{\mathbb{C}}(\Lambda))$ of $\text{KP}_{\mathbb{C}}(\Lambda)$. If Λ^0 is infinite, then $\text{KP}_{\mathbb{C}}(\Lambda) = \{0\}$.*

Proof. Let (p, s) be a generating Kumjian-Pask Λ -family for $\text{KP}_{\mathbb{C}}(\Lambda)$ and (q, t) a generating Cuntz-Krieger Λ -family for $C^*(\Lambda)$. Then (q, t) is a Kumjian-Pask Λ -family in $C^*(\Lambda)$, and the universal property of $\text{KP}_{\mathbb{C}}(\Lambda)$ gives a $*$ -homomorphism $\pi_{q,t} : \text{KP}_{\mathbb{C}}(\Lambda) \rightarrow C^*(\Lambda)$ which takes $s_\mu s_\nu^*$ to $t_\mu t_\nu^*$. It follows from the graded-uniqueness theorem that $\pi_{q,t}$ is a $*$ -isomorphism onto a dense $*$ -subalgebra of $C^*(\Lambda)$ (see Proposition 7.3 of [3]). Since Λ is aperiodic and cofinal, $C^*(\Lambda)$ is simple by [9, Theorem 3.1].

Now suppose that Λ^0 is finite. Then $\text{KP}_{\mathbb{C}}(\Lambda)$ has identity $1_{\text{KP}_{\mathbb{C}}(\Lambda)} = \sum_{v \in \Lambda^0} p_v$ and $C^*(\Lambda)$ has identity $1_{C^*(\Lambda)} = \sum_{v \in \Lambda^0} q_v$, and $\pi_{q,t}$ is unital. By Lemma 3.1, $Z(C^*(\Lambda)) = \{z1_{C^*(\Lambda)} : z \in \mathbb{C}\}$. By Lemma 3.2, $Z(\pi_{q,t}(\text{KP}_{\mathbb{C}}(\Lambda))) = Z(C^*(\Lambda)) \cap \pi_{q,t}(\text{KP}_{\mathbb{C}}(\Lambda)) = \{z1_{C^*(\Lambda)} : z \in \mathbb{C}\}$. Since $\pi_{q,t}$ is unital, $Z(\text{KP}_{\mathbb{C}}(\Lambda))$ is isomorphic to \mathbb{C} as claimed.

Next suppose that Λ^0 is infinite. Then $Z(C^*(\Lambda)) = \{0\}$ and $Z(\pi_{q,t}(\text{KP}_{\mathbb{C}}(\Lambda))) = Z(C^*(\Lambda)) \cap \pi_{q,t}(\text{KP}_{\mathbb{C}}(\Lambda)) = \{0\}$, giving $\text{KP}_{\mathbb{C}}(\Lambda) = \{0\}$. \square

4. THE CENTER OF A KUMJIAN-PASK ALGEBRA

Our goal is to extend Corollary 3.3 to Kumjian-Pask algebras over arbitrary rings. Throughout R is a commutative ring with 1 and Λ is a row-finite k -graph with no sources.

We will need Lemma 4.1 several times. For notational convenience, for $v \in \Lambda^0$, s_v or s_{v^*} means p_v . So when $m = 0$ Lemma 4.1, says that the set $\{s_\alpha : \alpha \in \Lambda\}$ is linearly independent.

Lemma 4.1. *Let $m \in \mathbb{N}^k$. Then $\{s_\alpha s_{\beta^*} : s(\alpha) = s(\beta) \text{ and } d(\beta) = m\}$ is a linearly independent subset of $\text{KP}_R(\Lambda)$.*

Proof. Let F be a finite subset of $\{(\alpha, \beta) \in \Lambda \times \Lambda^m : s(\alpha) = s(\beta)\}$, and suppose that $\sum_{(\alpha, \beta) \in F} r_{\alpha, \beta} s_\alpha s_{\beta^*} = 0$. Fix $(\sigma, \tau) \in F$. Since all the β have degree m , using (KP3) twice we obtain

$$\begin{aligned} 0 &= s_{\sigma^*} \left(\sum_{(\alpha, \beta) \in F} r_{\alpha, \beta} s_\alpha s_{\beta^*} \right) s_\tau \\ &= r_{\sigma, \tau} p_{s(\sigma)} + \sum_{(\alpha, \beta) \in F \setminus \{(\sigma, \tau)\}} r_{\alpha, \tau} s_{\sigma^*} s_\alpha s_{\beta^*} s_\tau \\ &= r_{\sigma, \tau} p_{s(\sigma)} + \sum_{\substack{(\alpha, \tau) \in F \\ \alpha \neq \sigma}} r_{\alpha, \tau} s_{\sigma^*} s_\alpha. \end{aligned} \tag{1}$$

If $d(\sigma) = d(\alpha)$ and $\sigma \neq \alpha$, then $s_{\sigma^*} s_\alpha = 0$ by (KP3). If $d(\sigma) \neq d(\alpha)$ then, by [3, Lemma 3.1], $s_{\sigma^*} s_\alpha$ is a linear combination of $s_\mu s_{\nu^*}$ where $d(\mu) - d(\nu) = d(\alpha) - d(\sigma)$. It follows that the 0-graded component of (1) is $r_{\sigma, \tau} p_{s(\sigma)}$. Thus $0 = r_{\sigma, \tau} p_{s(\sigma)}$. But $p_{s(\sigma)} \neq 0$ by Theorem 3.4 of [3]. Hence $r_{\sigma, \tau} = 0$. Since $(\sigma, \tau) \in F$ was arbitrary, it follows that $\{s_\alpha s_{\beta^*} : s(\alpha) = s(\beta) \text{ and } d(\beta) = m\}$ is linearly independent. \square

The next lemma describes properties of elements in the center of $\text{KP}_R(\Lambda)$.

Lemma 4.2. *Let $a \in Z(\text{KP}_R(\Lambda)) \setminus \{0\}$ be in normal form $\sum_{(\alpha, \beta) \in F} r_{\alpha, \beta} s_\alpha s_{\beta^*}$.*

- (1) *If $(\sigma, \tau) \in F$, then $r(\sigma) = r(\tau)$.*
- (2) *Let $W = \{v \in \Lambda^0 : \exists (\alpha, \beta) \in F \text{ with } v = r(\beta)\}$. If $\mu \in \Lambda W$, then $r(\mu) \in W$.*
- (3) *If $(\sigma, \tau) \in F$, then there exists $(\alpha, \beta) \in F$ such that $r(\alpha) = r(\beta) = s(\sigma) = s(\tau)$.*
- (4) *There exists $l \in \mathbb{N} \setminus \{0\}$ and $\{(\alpha_i, \beta_i)\}_{i=1}^l \subset F$ such that $\beta_1 \cdots \beta_l$ is a closed path in Λ .*

Proof. (1) Let $(\sigma, \tau) \in F$. By [5, Lemma 2.3] we have $0 \neq s_{\sigma^*} a s_\tau$. Since $a \in Z(\text{KP}_R(\Lambda))$

$$0 \neq s_{\sigma^*} p_{r(\sigma)} a p_{r(\tau)} s_\tau = s_{\sigma^*} a p_{r(\sigma)} p_{r(\tau)} s_\tau = \delta_{r(\sigma), r(\tau)} s_{\sigma^*} a s_\tau.$$

Hence $r(\sigma) = r(\tau)$.

(2) By way of contradiction, assume there exists $\mu \in \Lambda W$ such that $r(\mu) \notin W$. Then $p_v p_{r(\mu)} = 0$ for all $v \in W$. Thus

$$a p_{r(\mu)} = \sum_{(\alpha, \beta) \in F} r_{\alpha, \beta} s_\alpha s_{\beta^*} p_{r(\beta)} p_{r(\mu)} = 0.$$

Since $a \in Z(\text{KP}_R(\Lambda))$ we get $s_\mu a = a s_\mu = a p_{r(\mu)} s_\mu = 0$. Since $s(\mu) \in W$, there exist $(\alpha', \beta') \in F$ with $r(\beta') = s(\mu)$. Then $r(\alpha') = s(\mu)$ also by (1). Thus $S := \{(\alpha, \beta) \in F : s(\mu) = r(\alpha)\}$ is non-empty, and

$$0 = s_\mu a = \sum_{(\alpha, \beta) \in S} r_{\alpha, \beta} s_{\mu\alpha} s_{\beta^*}.$$

But $\{s_{\mu\alpha} s_{\beta^*} : (\alpha, \beta) \in S\}$ is linearly independent by Lemma 4.1, and hence $r_{\alpha, \beta} = 0$ for all $(\alpha, \beta) \in S$. This contradicts the given normal form for a .

(3) Let $(\sigma, \tau) \in F$. Then $s(\sigma) = s(\tau)$ by definition of normal form. By Lemma 2.3 in [5] we have $s_{\sigma^*} a s_{\tau} \neq 0$. Since $a \in Z(KP_R(\Lambda))$,

$$0 \neq s_{\sigma^*} a s_{\tau} = a s_{\sigma^*} s_{\tau} = \sum_{(\alpha, \beta) \in F} r_{\alpha, \beta} s_{\alpha} s_{\beta^*} s_{\sigma^*} s_{\tau} = \sum_{\substack{(\alpha, \beta) \in F \\ r(\beta) = s(\sigma)}} r_{\alpha, \beta} s_{\alpha} s_{(\sigma\beta)^*} s_{\tau}. \quad (2)$$

In particular, the set $\{(\alpha, \beta) \in F : r(\beta) = s(\sigma)\}$ is nonempty. So there exists $(\alpha', \beta') \in F$ such that $r(\beta') = s(\sigma)$. Since $r(\alpha') = r(\beta')$ from (1), we are done.

(4) Let $M = |F| + 1$. Using (3) there exists a path $\beta_1 \dots \beta_M$ such that, for $1 \leq i \leq M$, there exists $\alpha_i \in \Lambda$ with $(\alpha_i, \beta_i) \in F$. Since $M > |F|$, there exists $i < j \in \{1, \dots, M\}$ such that $\beta_i = \beta_j$. Then $\beta_i \dots \beta_{j-1}$ is a closed path. \square

The next corollary follows from Lemma 4.2(4).

Corollary 4.3. *Let Λ be a row-finite k -graph with no sources and R a commutative ring with 1. If Λ has no closed paths then the center $Z(KP_R(\Lambda)) = \{0\}$.*

The next lemma provides a description of elements of the center of $KP_R(\Lambda)$ when Λ is cofinal.

Lemma 4.4. *Suppose that Λ is cofinal. Let $a \in Z(KP_R(\Lambda)) \setminus \{0\}$ be in normal form $\sum_{(\alpha, \beta) \in F} r_{\alpha, \beta} s_{\alpha} s_{\beta^*}$. Then $\{v \in \Lambda^0 : \exists(\alpha, \beta) \in F \text{ with } v = r(\beta)\} = \Lambda^0$.*

Proof. Write $W := \{v \in \Lambda^0 : \exists(\alpha, \beta) \in F \text{ with } v = r(\beta)\}$. Since $\sum_{(\alpha, \beta) \in F} r_{\alpha, \beta} s_{\alpha} s_{\beta^*}$ is in normal form, $F \subset \Lambda \times \Lambda^m$ for some $m \in \mathbb{N}^k$. Let $n = m \vee (1, 1, \dots, 1)$. By (KP4), for each $(\alpha, \beta) \in F$, we have $s_{\alpha} s_{\beta^*} = \sum_{\mu \in s(\alpha) \Lambda^{n-m}} s_{(\alpha\mu)} s_{(\beta\mu)^*}$. By “reshaping” each pair of paths in F in this way, collecting like terms and dropping those with zero coefficients, we see that there exists $G \subset \Lambda \times \Lambda^n$ and $r'_{\gamma, \eta} \in R \setminus \{0\}$ such that $a = \sum_{(\gamma, \eta) \in G} r'_{\gamma, \eta} s_{\gamma} s_{\eta^*}$ is also in normal form. By construction, $W' = \{v \in \Lambda^0 : \exists(\gamma, \eta) \in G \text{ with } v = r(\eta)\} \subset W$.

Let $v \in \Lambda^0$. Using Lemma 4.2(4), there exists $\{(\gamma_i, \eta_i)\}_{i=1}^l \subset G$ such that $\eta_1 \dots \eta_l$ is a closed path. Since $d(\eta_i) \geq (1, 1, \dots, 1)$ for all i , $x := \eta_1 \dots \eta_l \eta_1 \dots \eta_l$ is an infinite path. By cofinality, there exist $q \in \mathbb{N}^k$ and $\nu \in v \Lambda x(q)$. By the definition of x , there exist $q' \geq q$ and j such that $x(q') = r(\eta_j)$. Let $\lambda = x(q, q')$. Then $\nu \lambda \in v \Lambda W'$. By Lemma 4.2(2), $v = r(\nu \lambda) \in W'$ as well. Thus $W' = \Lambda^0$, and since $W' \subset W$ we have $W = \Lambda^0$. \square

The next lemma provides a description of elements of the center of $KP_R(\Lambda)$ when Λ is aperiodic.

Lemma 4.5. *Suppose that Λ is aperiodic and $a \in Z(KP_R(\Lambda)) \setminus \{0\}$. Then there exist $n \in \mathbb{N}^k$ and $G \subset \Lambda^n$ such that $a = \sum_{\alpha \in G} r_{\alpha} s_{\alpha} s_{\alpha^*}$ is in normal form.*

Proof. Suppose $a \in Z(KP_R(\Lambda)) \setminus \{0\}$ with $a = \sum_{(\alpha, \beta) \in F} r_{\alpha, \beta} s_{\alpha} s_{\beta^*}$ in normal form. Let $(\sigma, \tau) \in F$. From Lemma 2.3 in [5] we know that $s_{\sigma^*} a s_{\tau} \neq 0$. Let $m = \vee_{(\alpha, \beta) \in F} (d(\alpha) \vee d(\beta))$. Since Λ is aperiodic, by [6, Lemma 6.2], there exist $\lambda \in s(\sigma) \Lambda$ with $d(\lambda) \geq m$ such that

$$\left. \begin{array}{l} \alpha, \beta \in \Lambda s(\sigma), \ d(\alpha), d(\beta) \leq m \\ \text{and } \alpha \lambda(0, d(\lambda)) = \beta \lambda(0, d(\lambda)) \end{array} \right\} \implies \alpha = \beta. \quad (3)$$

The same argument as in [3, Proposition 4.9] now shows that $s_{\lambda^*} s_{\sigma^*} a s_{\tau} s_{\lambda} \neq 0$. Since $a \in Z(KP_R(\Lambda))$, $0 \neq s_{\lambda^*} s_{\sigma^*} a s_{\tau} s_{\lambda} = a s_{\lambda^*} s_{\sigma^*} s_{\tau} s_{\lambda} = a s_{(\sigma\lambda)^*} s_{\tau\lambda}$. Thus

$$\begin{aligned} 0 \neq s_{(\sigma\lambda)^*} s_{\tau\lambda} &= s_{\sigma\lambda(d(\lambda), d(\lambda)+d(\sigma))^*} s_{\sigma\lambda(0, d(\lambda))^*} s_{\tau\lambda(0, d(\lambda))} s_{\tau\lambda(d(\lambda), d(\lambda)+d(\tau))} \\ &= \delta_{\sigma, \tau} s_{\sigma\lambda(d(\lambda), d(\lambda)+d(\sigma))^*} s_{\tau\lambda(d(\lambda), d(\lambda)+d(\tau))} \quad \text{using (3).} \end{aligned}$$

Thus $\sigma = \tau$.

Since $(\sigma, \tau) \in F$ was arbitrary we have $\alpha = \beta$ for all $(\alpha, \beta) \in F$. Let $G = \{\alpha \in \Lambda : (\alpha, \alpha) \in F\}$ and write r_α for $r_{\alpha, \alpha}$. Note $G \subset \Lambda^n$ because $F \subset \Lambda \times \Lambda^n$ for some. Thus $a = \sum_{\alpha \in G} r_\alpha s_\alpha s_{\alpha^*}$ in normal form as desired. \square

Our main theorem (Theorem 4.7) has two cases: Λ^0 finite and infinite.

Lemma 4.6. *$\text{KP}_R(\Lambda)$ has an identity if and only if Λ^0 is finite.*

Proof. If Λ^0 is finite, then $\sum_{v \in \Lambda^0} p_v$ is an identity for $\text{KP}_R(\Lambda)$. Conversely, assume that $\text{KP}_R(\Lambda)$ has an identity $1_{\text{KP}_R(\Lambda)}$. By way of contradiction, suppose that Λ^0 is infinite. Write $1_{\text{KP}_R(\Lambda)}$ in normal form $\sum_{(\alpha, \beta) \in F} r_{\alpha, \beta} s_\alpha s_{\beta^*}$. Since F is finite, so is $W := \{v \in \Lambda^0 : \exists (\alpha, \beta) \in F \text{ with } v = r(\beta)\}$. Thus there exists $w \in \Lambda^0 \setminus W$. Now $p_w = 1_{\text{KP}_R(\Lambda)} p_w = \sum_{(\alpha, \beta) \in F} r_{\alpha, \beta} s_\alpha s_{\beta^*} p_w = 0$ because $w \neq r(\beta)$ for any of the β . This contradiction shows that Λ^0 must be finite. \square

Theorem 4.7. *Let Λ be a row-finite k -graph with no sources and R a commutative ring with 1.*

- (1) *Suppose Λ is aperiodic and cofinal, and that Λ^0 is finite. Then $Z(\text{KP}_R(\Lambda)) = R1_{\text{KP}_R(\Lambda)}$.*
- (2) *Suppose that Λ is cofinal and Λ^0 is infinite. Then $Z(\text{KP}_R(\Lambda)) = \{0\}$.*

Proof. (1) Suppose Λ is aperiodic and cofinal, and that Λ^0 is finite. Let $a \in Z(\text{KP}_R(\Lambda)) \setminus \{0\}$. Since Λ is aperiodic, by Lemma 4.5, there exist $G \subset \Lambda^n$ such that $a = \sum_{\alpha \in G} r_\alpha s_\alpha s_{\alpha^*}$ is in normal form. Since Λ is row-finite and Λ^0 is finite, Λ^n is finite.

We claim that $G = \Lambda^n$. By way of contradiction, suppose that $G \neq \Lambda^n$, and let $\lambda \in \Lambda^n \setminus G$. Then $as_\lambda = 0$ by (KP3). But since $a \in Z(\text{KP}_R(\Lambda))$,

$$0 = as_\lambda = s_\lambda a = \sum_{\substack{\alpha \in G \\ r(\alpha) = s(\lambda)}} r_\alpha s_{\lambda\alpha} s_{\alpha^*}.$$

Since Λ is cofinal, $\{r(\alpha) : \alpha \in G\} = \Lambda^0$ by Lemma 4.4. Thus $S = \{\alpha \in G : r(\alpha) = s(\lambda)\} \neq \emptyset$. But $\{s_{\lambda\alpha} s_{\alpha^*} : \alpha \in S\}$ is linearly independent by Lemma 4.1. Thus $r_\alpha = 0$ for $\alpha \in S$, contradicting our choice of $\{r_\alpha\}$. It follows that $G = \Lambda^n$ as claimed, and that

$$a = \sum_{\alpha \in \Lambda^n} r_\alpha s_\alpha s_{\alpha^*}.$$

Next we claim that $r_\mu = r_\nu$ for all $\mu, \nu \in \Lambda^n$. Let $\mu, \nu \in \Lambda^n$. Let $x \in s(\mu)\Lambda^\infty$. Since Λ is cofinal, there exists $m \in \mathbb{N}^k$ and $\gamma \in s(\nu)\Lambda s(x(m))$. Set $\eta = x(0, m)$. Now

$$\begin{aligned} r_\mu s_{\nu\gamma} s_{(\mu\eta)^*} &= r_\mu s_{\nu\gamma} s_{\eta^*} s_{\mu^*} = s_{\nu\gamma} s_{\eta^*} \sum_{\alpha \in \Lambda^n} r_\alpha s_\alpha s_{\alpha^*} \\ &= s_{\nu\gamma} s_{\eta^*} s_{\mu^*} \sum_{\alpha \in \Lambda^n} r_\alpha s_\alpha s_{\alpha^*} = s_{\nu\gamma} s_{(\mu\eta)^*} a \\ &= a s_{\nu\gamma} s_{(\mu\eta)^*} = \sum_{\alpha \in \Lambda^n} r_\alpha s_\alpha s_{\alpha^*} s_{\nu\gamma} s_{(\mu\eta)^*} = r_\nu s_{\nu\gamma} s_{(\mu\eta)^*}. \end{aligned}$$

Since $s_{\nu\gamma} s_{(\mu\eta)^*} \neq 0$ this implies $r_\mu = r_\nu$. Let $r = r_\mu$. Now

$$a = \sum_{\alpha \in \Lambda^n} r s_\alpha s_{\alpha^*} = \sum_{v \in \Lambda^0} \sum_{\alpha \in v\Lambda^n} r s_\alpha s_{\alpha^*} = r \sum_{v \in \Lambda^0} p_v = r 1_{\text{KP}_R(\Lambda)}$$

as desired.

(2) Suppose there exists $a \in Z(KP_R(\Lambda)) \setminus \{0\}$. Write $a = \sum_{(\alpha, \beta) \in F} r_{\alpha, \beta} s_{\alpha} s_{\beta}^*$ in normal form. Then Lemma 4.4 gives that $\{v \in \Lambda^0 : \exists (\alpha, \beta) \in F \text{ with } v = r(\beta)\} = \Lambda^0$, contradicting that F is finite. \square

Simplicity of $C^*(\Lambda)$ played an important role in §3. To reconcile this with Theorem 4.7, recall from [10] that an ideal $I \in KP_R(\Lambda)$ is *basic* if $rp_v \in I$ for $r \in R \setminus \{0\}$ then $p_v \in I$, and that $KP_R(\Lambda)$ is *basically simple* if its only basic ideals are $\{0\}$ and $KP_R(\Lambda)$. By [3, Theorem 5.14], $KP_R(\Lambda)$ is basically simple if and only if Λ is cofinal and aperiodic (and by [3, Theorem 6.1], $KP_R(\Lambda)$ is simple if and only if R is a field and Λ is cofinal and aperiodic). Thus Theorem 4.7 is in the spirit of Corollary 3.3.

5. COMMUTATIVE KUMJIAN-PASK ALGEBRAS

We view \mathbb{N}^k as a category with one object \star and composition given by addition, and use $\{e_i\}_{i=1}^k$ to denote the standard basis of \mathbb{N}^k .

Example 5.1. Let $d : \mathbb{N}^k \rightarrow \mathbb{N}^k$ be the identity map. Then (\mathbb{N}^k, d) is a k -graph. By [3, Example 7.1], $KP_R(\mathbb{N}^k)$ is commutative with identity p_{\star} , and $KP_R(\mathbb{N}^k)$ is isomorphic to the ring of Laurent polynomials $R[x_1, x_1^{-1}, \dots, x_k, x_k^{-1}]$ in k commuting indeterminates.

Lemma 5.2. *Suppose $\Lambda = \Lambda_1 \sqcup \Lambda_2$ is a disjoint union of two k -graphs. Then $KP_R(\Lambda) = KP_R(\Lambda_1) \oplus KP_R(\Lambda_2)$.*

Proof. For each $i = 1, 2$, let (q^i, t^i) be the generating Kumjian-Pask Λ_i -family of $KP_R(\Lambda_i)$, and let (p, s) be the generating Kumjian-Pask Λ -family of $KP_R(\Lambda)$. Restricting (p, s) to Λ_i gives a Λ_i -family in $KP_R(\Lambda)$, and hence the universal property for $KP_R(\Lambda_i)$ gives a homomorphism $\pi_{p, s}^i : KP_R(\Lambda_i) \rightarrow KP_R(\Lambda)$ such that $\pi_{p, s}^i \circ (q^i, t^i) = (p, s)$. Each $\pi_{p, s}^i$ is graded, and the graded uniqueness theorem ([3, Theorem 4.1]) implies that $\pi_{p, s}^i$ is injective.

We now identify $KP_R(\Lambda_i)$ with its image in $KP_R(\Lambda)$. If $\mu \in \Lambda_1$ and $\lambda \in \Lambda_2$, then $s_{\mu} s_{\lambda} = s_{\mu} p_{s(\mu)} p_{r(\lambda)} s_{\lambda} = 0$. Similarly $s_{\lambda} s_{\mu}, s_{\mu}^* s_{\lambda}^*, s_{\lambda}^* s_{\mu}^*, s_{\lambda} s_{\mu}^*, s_{\mu}^* s_{\lambda}, s_{\lambda}^* s_{\mu}, s_{\mu} s_{\lambda}^*$ are all zero. Thus $KP_R(\Lambda_1) KP_R(\Lambda_2) = 0 = KP_R(\Lambda_2) KP_R(\Lambda_1)$, and the internal direct sum $KP_R(\Lambda_1) \oplus KP_R(\Lambda_2)$ is a subalgebra of $KP_R(\Lambda)$. Finally, $KP_R(\Lambda_1) \oplus KP_R(\Lambda_2)$ is all of $KP_R(\Lambda)$ since the former contains all the generators of later. This gives the result. \square

Proposition 5.3. *Let Λ be a row-finite k -graph with no sources and R a commutative ring with 1. Then the following conditions are equivalent:*

- (1) $KP_R(\Lambda)$ is commutative;
- (2) $r = s$ on Λ and $r|_{\Lambda^n}$ is injective;
- (3) $\Lambda \cong \bigsqcup_{v \in \Lambda^0} \mathbb{N}^k$;
- (4) $KP_R(\Lambda) \cong \bigoplus_{v \in \Lambda^0} R[x_1, x_1^{-1}, \dots, x_k, x_k^{-1}]$.

Proof. (1) \Rightarrow (2). Suppose that $KP_R(\Lambda)$ is commutative. By way of contradiction, suppose there exists $\lambda \in \Lambda$ such that $s(\lambda) \neq r(\lambda)$. Then $s_{\lambda}^* s_{\lambda} = s_{\lambda} s_{\lambda}^*$, and

$$p_{s(\lambda)} = p_{s(\lambda)}^2 = p_{s(\lambda)} s_{\lambda}^* s_{\lambda} = p_{s(\lambda)} s_{\lambda} s_{\lambda}^* = p_{s(\lambda)} p_{r(\lambda)} s_{\lambda} s_{\lambda}^* = 0.$$

But $p_v \neq 0$ for all $v \in \Lambda^0$ by [3, Theorem 3.4]. This contradiction gives $r = s$.

Next, suppose $\lambda, \mu \in \Lambda^n$ with $\lambda \neq \mu$. By way of contradiction, suppose that $r(\lambda) = r(\mu)$. Since $r = s$, $r(\lambda) = s(\lambda) = s(\mu) = r(\mu)$. Then

$$s_{\lambda} = p_{r(\lambda)} s_{\lambda} = p_{s(\lambda)} s_{\lambda} = p_{s(\mu)} s_{\lambda} = s_{\mu}^* s_{\mu} s_{\lambda} = s_{\mu}^* s_{\lambda} s_{\mu} = 0$$

by (KP3). Now $p_{s(\lambda)} = 0$, contradicting that $p_v \neq 0$ for all $v \in \Lambda^0$ by [3, Theorem 3.4]. Thus r is injective on Λ^n .

(2) \Rightarrow (3) Assume that $r = s$ on Λ and that $r|_{\Lambda^n}$ is injective. Since $r = s$, $\{v\Lambda v\}_{v \in \Lambda^0}$ is a partition of Λ . Since r is injective on Λ^{e_i} , the subgraph $v\Lambda^{e_i}v$ has a single vertex v and single edge f_i^v . Thus $f_i^v \mapsto e_i$ defines a graph isomorphism $v\Lambda v \rightarrow \mathbb{N}^k$. Hence $\Lambda = \bigsqcup_{v \in \Lambda^0} v\Lambda v \cong \bigsqcup_{v \in \Lambda^0} \mathbb{N}^k$.

(3) \Rightarrow (4) Assume that $\Lambda \cong \bigsqcup_{v \in \Lambda^0} \mathbb{N}^k$. By Lemma 5.2, $\text{KP}_R(\Lambda)$ is isomorphic to $\bigoplus \text{KP}_R(\mathbb{N}^k)$, and by Example 5.1 each $\text{KP}_R(\mathbb{N}^k)$ is isomorphic to $R[x_1, x_1^{-1}, \dots, x_k, x_k^{-1}]$.

(4) \Rightarrow (1) Follows since $\bigoplus R[x_1, x_1^{-1}, \dots, x_k, x_k^{-1}]$ is commutative. \square

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